Exercise 37

If a wire with linear density $\rho(x, y)$ lies along a plane curve C, its **moments of inertia** about the x- and y-axes are defined as

$$I_x = \int_C y^2 \rho(x, y)\, ds \qquad I_y = \int_C x^2 \rho(x, y)\, ds$$

Find the moments of inertia for the wire in Example 3.

Solution

The wire in Example 3 is the semicircle in the upper half-plane, $x^2 + y^2 = 1,\ y \geq 0$, with the linear density

$$\rho(x, y) = k(1 - y).$$

In order to evaluate the line integrals, parameterize the semicircle by $x = \cos t$ and $y = \sin t$, where $0 \leq t \leq \pi$. Calculate I_x first.

$$\begin{aligned}
I_x &= \int_C y^2 \rho(x, y)\, ds \\
&= \int_0^\pi [y(t)]^2 k[1 - y(t)] \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= k \int_0^\pi \sin^2 t(1 - \sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \\
&= k \int_0^\pi \sin^2 t(1 - \sin t)\sqrt{1}\, dt \\
&= k \left(\int_0^\pi \sin^2 t\, dt - \int_0^\pi \sin^2 t \cdot \sin t\, dt\right) \\
&= k \left[\int_0^\pi \frac{1}{2}(1 - \cos 2t)\, dt - \int_0^\pi (1 - \cos^2 t) \sin t\, dt\right]
\end{aligned}$$

Make the following substitution in the second integral.

$$\begin{aligned}
u &= \cos t \\
du &= -\sin t\, dt \quad \rightarrow \quad -du = \sin t\, dt
\end{aligned}$$

Consequently,

$$\begin{aligned}
I_x &= k\left[\frac{1}{2}\left(\int_0^\pi dt - \int_0^\pi \cos 2t\, dt\right) - \int_{\cos 0}^{\cos \pi} (1 - u^2)\,(-du)\right] \\
&= k\left[\frac{1}{2}\left(\pi - \underbrace{\frac{1}{2}\sin 2t\Big|_0^\pi}_{=0}\right) - \int_1^{-1} (1 - u^2)\,(-du)\right].
\end{aligned}$$

Evaluate the remaining integral.

$$\begin{aligned} I_x &= k\left[\frac{\pi}{2} - \int_{-1}^{1}(1-u^2)\,du\right] \\ &= k\left[\frac{\pi}{2} - 2\int_{0}^{1}(1-u^2)\,du\right] \\ &= k\left[\frac{\pi}{2} - 2\left(u - \frac{u^3}{3}\right)\bigg|_0^1\right] \\ &= k\left[\frac{\pi}{2} - 2\left(1 - \frac{1}{3}\right)\right] \end{aligned}$$

Therefore,

$$\boxed{I_x = k\left(\frac{\pi}{2} - \frac{4}{3}\right).}$$

Calculate I_y now.

$$\begin{aligned} I_y &= \int_C x^2 \rho(x,y)\,ds \\ &= \int_0^{\pi} [x(t)]^2 k[1-y(t)]\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= k\int_0^{\pi} \cos^2 t(1-\sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt \\ &= k\int_0^{\pi} \cos^2 t(1-\sin t)\sqrt{1}\,dt \\ &= k\left(\int_0^{\pi} \cos^2 t\,dt - \int_0^{\pi} \cos^2 t \sin t\,dt\right) \\ &= k\left[\int_0^{\pi} \frac{1}{2}(1+\cos 2t)\,dt - \int_0^{\pi} \cos^2 t \sin t\,dt\right] \end{aligned}$$

Make the following substitution in the second integral.

$$\begin{aligned} u &= \cos t \\ du &= -\sin t\,dt \quad \rightarrow \quad -du = \sin t\,dt \end{aligned}$$

Consequently,

$$\begin{aligned} I_y &= k\left[\frac{1}{2}\left(\int_0^{\pi} dt + \int_0^{\pi} \cos 2t\,dt\right) - \int_{\cos 0}^{\cos \pi} u^2\,(-du)\right] \\ &= k\left[\frac{1}{2}\left(\pi + \underbrace{\frac{1}{2}\sin 2t\bigg|_0^{\pi}}_{=0}\right) - \int_1^{-1} u^2\,(-du)\right]. \end{aligned}$$

Evaluate the remaining integral.

$$\begin{aligned} I_y &= k\left(\frac{\pi}{2} - \int_{-1}^{1} u^2\,du\right) \\ &= k\left(\frac{\pi}{2} - 2\int_{0}^{1} u^2\,du\right) \\ &= k\left[\frac{\pi}{2} - 2\left(\frac{u^3}{3}\right)\Bigg|_0^1\right] \\ &= k\left[\frac{\pi}{2} - 2\left(\frac{1}{3}\right)\right] \end{aligned}$$

Therefore,

$$\boxed{I_y = k\left(\frac{\pi}{2} - \frac{2}{3}\right).}$$